Renormalization in Dynamical Systems

Nonlinear and Complex Systems Group Research Programme

Families of quadratic maps of the form

$$x_{n+1} = f_\lambda(x_n) = \lambda x_n(1 - x_n),$$

display a period-doubling cascade along the route to chaos which possesses remarkable universal quantitative features. This universality was explained, in renormalization terms, by Feigenbaum and others in the late 1970s.

The quantitative universality is determined by the stability properties of a critical fixed point (function) of an operator acting on a certain space of functions. In particular, the universal constants observed are given in terms of the fixed point and the eigenvalues of the linearised operator there. Such exponents are shared by all maps in a broad *universality class*, characterised by the degree of the turning point of the maps, f_λ.

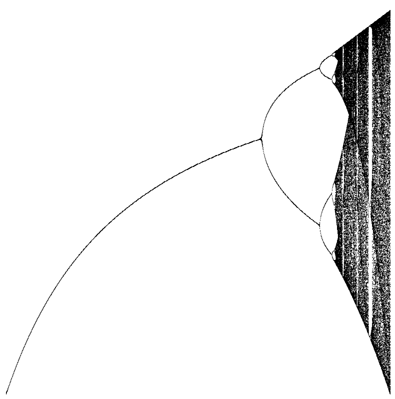

Figure: bifurcation diagram showing the period-doubling route to chaos in a one-parameter family of quadratic maps.

Similar ideas are used to understand period-doubling in Hamiltonian systems, with critical functions and exponents lying in a different universality class.

A different route to chaos is via the breakup of quasiperiodic orbits in dynamical systems including dissipative systems (exemplified by circle maps), Hamiltonian systems (KAM tori), and the iteration of complex analytic maps (Siegel discs).

The universality observed in each of these scenarios has a similar explanation in terms of a renormalization operator acting on an appropriate space.

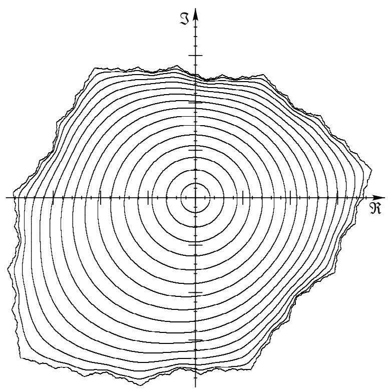

Figure: Siegel disc for a quadratic complex map

Although analytic proofs for the existence of renormalization fixed points (and the eigenstructure of the corresponding operator there) are possible in certain cases, such results are difficult to come by. Instead, a number of these questions have been settled via rigorous computer-assisted proofs, which extend the concept of interval arithmetic to the algebra of operators on the appropriate spaces.

More recent studies focus on period doubling in coupled systems and universality in quasiperiodically-driven systems.

The Nonlinear and Complex Systems Group welcomes enquiries regarding job vacancies, Ph.D. and Postdoctoral study, and academic and industrial collaboration on its research programmes.

For further details, contact:

Nonlinear and Complex Systems Group
Department of Mathematics, University of Portsmouth
Lion Terrace, Portsmouth PO1 3HF, United Kingdom
t: +44 (0)23 9284 6367 e: hod.maths@port.ac.uk
f: +44 (0)23 9284 6365 w: www.port.ac.uk/maths